Forecasting age distribution of life-table death counts via α-transformation

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3. Compositional power α-transformation to model & forecast life-table death counts

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$$S^D = \left\{ (d_1, d_2, \dots, d_D)^\top, \quad d_x > 0, \quad \sum_{x=1}^{D} d_x = c \right\}$$

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4. Oeppen (2008) and Bergeron-Boucher et al. (2017) combine CoDa with PCA to model & forecast life-table death counts

clr transformation

1. Transform compositional data from simplex to Euclidean space

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2. Extending clr, isometric log-ratio (ilr) is

$$\boldsymbol{z} = \boldsymbol{H}\boldsymbol{s}$$

where $\boldsymbol{H}$: $(D-1)\times D$: Helmert matrix

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3. To remove any redundant dimension

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1. No guarantee transformed data are multivariate normal distributed

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2. Cannot be applied to data containing zeros

Australian data

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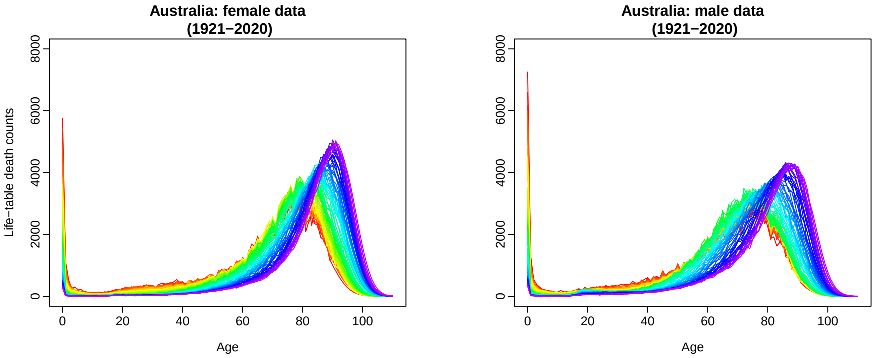
3. 111 ages, ages $0, 1, \dots, 109, 110+$

Rainbow plot



1. Decreasing trend in infant mortality
2. A negatively skewed distribution for life-table death counts
3. Shift of the distribution symbolises longevity risk

α transformation

1. α-transformation of a compositional vector $\boldsymbol{d}_t \in \mathcal{S}^D$

$$\boldsymbol{z}_t^{\alpha} = \boldsymbol{H} \cdot \left(\frac{D\boldsymbol{d}_t^{\alpha} - \mathbb{1}_D}{\alpha} \right)$$

$$\boldsymbol{d}_t^{\alpha} = \left(\frac{d_{t,1}^{\alpha}}{\sum_{j=1}^{D} d_{t,j}^{\alpha}}, \dots, \frac{d_{t,D}^{\alpha}}{\sum_{j=1}^{D} d_{t,j}^{\alpha}} \right)^{\top},$$

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2. α transformation is invertible

$$\boldsymbol{v}_t^\alpha = D\boldsymbol{d}_t^\alpha = \alpha \times \boldsymbol{H}^\top \boldsymbol{z}_t^\alpha + \mathbb{1}_D$$

$$\boldsymbol{d}_t = \left(\frac{v_{t,1}^{1/\alpha}}{\sum_{j=1}^D v_{t,j}^{1/\alpha}}, \dots, \frac{v_{t,D}^{1/\alpha}}{\sum_{j=1}^D v_{t,j}^{1/\alpha}} \right)$$

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- $\boldsymbol{e}_t$: model error term

Eigenvalue ratio criterion

1. Determine optimal number of components

$$K = \mathop{\arg\min}_{1\leq k\leq K_{\max}} \left\{ \frac{\widehat{\lambda}_{k+1}}{\widehat{\lambda}_k} \times \mathbb{1}\left\{\frac{\widehat{\lambda}_k}{\widehat{\lambda}_1} \geq \theta\right\} + \mathbb{1}\left\{\frac{\widehat{\lambda}_k}{\widehat{\lambda}_1} < \theta\right\} \right\}$$

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$$\widehat{\boldsymbol{z}}^{\alpha}_{n+h|n} = E[\boldsymbol{z}^{\alpha}_{n+h}|\widehat{\boldsymbol{\Phi}}, \boldsymbol{z}^{\alpha}] = \overline{z}^{\alpha} + \sum_{k=1}^{K} \widehat{\beta}_{n+h|n,k}\widehat{\phi}_k$$

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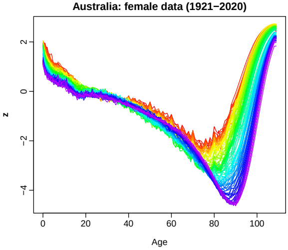
- $\widehat{\boldsymbol{\Phi}} = (\widehat{\Phi}_1, \dots, \widehat{\Phi}_K)^\top$
- $\widehat{\beta}_{n+h|n,k}$: h-step-ahead forecast of k^{th} principal component scores

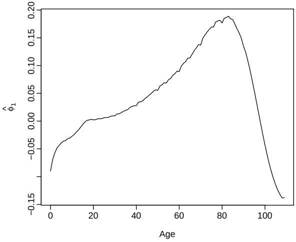
Selection of α parameter

Split data into a training set, a validation set, a testing set

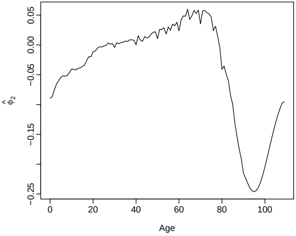
$1:(n-20)$	$(n-19):(n-10)$	$(n-9):n$
Training	Validation	Testing



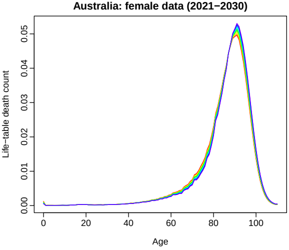
(a) *Unconstrained female data*



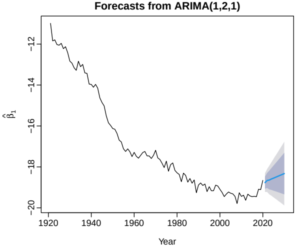
(b) *First principal component*



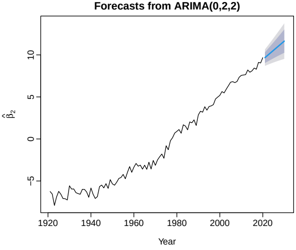
(c) *Second principal component*



(d) *Forecast LTDC*



(e) *Forecast first set of scores*



(f) *Forecast second set of scores*

Parameter estimation

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3. Multiplying forecast scores with estimated functional principal components to obtain forecast curves

Goodness-of-fit (R^2 criterion)

1. In-sample goodness-of-fit via an R^2 criterion

$$R^2 = 1 - \frac{\sum_{x=1}^{111}\sum_{t=1}^{100}(d_{t,x} - \widehat{d}_{t,x})^2}{\sum_{x=1}^{111}\sum_{t=1}^{100}(d_{x,t} - \overline{d}_x)^2}$$

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- $d_{t,x}$: observed age-specific life-table death counts

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2. Compared to ilr,

Method	Female	Male
ilr	0.9953	0.9911
α transformation	0.9968	0.9915

Point forecast error measure

1. Kullback-Leibler divergence is symmetric & non-negative:

$$\begin{aligned}\mathrm{KLD}(h) &= D_{\mathsf{KL}}\left(\boldsymbol{d}_{m+\xi}||\widehat{\boldsymbol{d}}_{m+\xi}\right) + D_{\mathsf{KL}}\left(\widehat{\boldsymbol{d}}_{m+\xi}||\boldsymbol{d}_{m+\xi}\right) \\ &= \frac{1}{111\times(11-h)}\sum_{\xi=h}^{10}\sum_{x=1}^{111} d_{m+\xi,x}\cdot\left(\ln d_{m+\xi,x} - \ln \widehat{d}_{m+\xi,x}\right) \\ &\quad + \frac{1}{111\times(11-h)}\sum_{\xi=h}^{10}\sum_{x=1}^{111} \widehat{d}_{m+\xi,x}\cdot\left(\ln \widehat{d}_{m+\xi,x} - \ln d_{m+\xi,x}\right)\end{aligned}$$

m: end of training or validation set

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$$+ \frac{1}{111\times(11-h)}\sum_{\xi=h}^{10}\sum_{x=1}^{111} \widehat{d}_{m+\xi,x}\cdot\left(\ln \widehat{d}_{m+\xi,x} - \ln d_{m+\xi,x}\right)$$

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2. Jensen-Shannon divergence:

$$\text{JSD}(h) = \frac{1}{2}D_{\text{KL}}\left(\boldsymbol{d}_{m+\xi}||\boldsymbol{\delta}_{n+\xi}\right) + \frac{1}{2}D_{\text{KL}}\left(\widehat{\boldsymbol{d}}_{m+\xi}||\boldsymbol{\delta}_{n+\xi}\right)$$

where $\boldsymbol{\delta}_{m+\xi}$ measures a common quantity between $\boldsymbol{d}_{m+\xi}$ & $\widehat{\boldsymbol{d}}_{m+\xi}$;

$$\boldsymbol{\delta}_{m+\xi} = \sqrt{\boldsymbol{d}_{m+\xi}\widehat{\boldsymbol{d}}_{m+\xi}}$$

KLD results

	Female			Male		
h	α-transform	ilr	eda	α-transform	ilr	eda
1	0.0038	0.0078	0.0828	0.0083	0.0107	0.0064
2	0.0040	0.0082	0.0842	0.0106	0.0121	0.0102
3	0.0047	0.0087	0.0853	0.0109	0.0137	0.0175
4	0.0051	0.0087	0.0868	0.0118	0.0151	0.0268
5	0.0054	0.0093	0.0889	0.0152	0.0172	0.0376
6	0.0060	0.0100	0.0928	0.0174	0.0197	0.0511
7	0.0063	0.0101	0.0959	0.0179	0.0230	0.0555
8	0.0082	0.0116	0.0966	0.0246	0.0277	0.0548
9	0.0083	0.0115	0.1043	0.0266	0.0298	0.0786
10	0.0107	0.0132	0.1076	0.0309	0.0345	0.1407
Mean	0.0062	0.0099	0.0925	0.0174	0.0204	0.0479

JSD(geo)

	Female			Male		
h	α-transform	ilr	eda	α-transform	ilr	eda
1	0.0009	0.0019	0.0099	0.0021	0.0027	0.0015
2	0.0010	0.0020	0.0104	0.0024	0.0030	0.0025
3	0.0012	0.0022	0.0108	0.0027	0.0034	0.0044
4	0.0015	0.0022	0.0115	0.0029	0.0038	0.0069
5	0.0013	0.0023	0.0123	0.0033	0.0043	0.0096
6	0.0016	0.0025	0.0133	0.0038	0.0049	0.0131
7	0.0014	0.0025	0.0141	0.0045	0.0057	0.0143
8	0.0024	0.0029	0.0145	0.0061	0.0069	0.0140
9	0.0020	0.0028	0.0165	0.0066	0.0074	0.0203
10	0.0029	0.0033	0.0176	0.0077	0.0086	0.0375
Mean	0.0016	0.0025	0.0131	0.0042	0.0051	0.0124

Interval score

Scoring rule for interval forecasts at time point $d_{n+\xi,x}$:

$$S_{\gamma,\xi}(\widehat{d}^{\text{lb}}_{m+\xi,x}, \widehat{d}^{\text{ub}}_{m+\xi,x}, d_{m+\xi,x}) = (\widehat{d}^{\text{ub}}_{m+\xi,x} - \widehat{d}^{\text{lb}}_{m+\xi,x}) + \frac{2}{\gamma}(\widehat{d}^{\text{lb}}_{m+\xi,x} - d_{m+\xi,x})\mathbb{1}\{d_{m+\xi,x} < \widehat{d}^{\text{lb}}_{m+\xi,x}\} + \frac{2}{\gamma}(d_{m+\xi,x} - \widehat{d}^{\text{ub}}_{m+\xi,x})\mathbb{1}\{d_{m+\xi,x} > \widehat{d}^{\text{ub}}_{m+\xi,x}\}$$

- γ: level of significance, customarily $\gamma = 0.2$
- For different ages & years in the forecasting period

$$\overline{S}_{\gamma}(h) = \frac{1}{111 \times (11-h)} \sum_{\xi=h}^{10} \sum_{x=1}^{111} S_{\gamma,\xi}\left(\widehat{d}^{\text{lb}}_{m+\xi,x}, \widehat{d}^{\text{ub}}_{m+\xi,x}; d_{m+\xi,x}\right)$$

CPD=|Nominal coverage - Empirical coverage|

	Female			Male		
h	α-transform	ilr	eda	α-transform	ilr	eda
1	0.0623	0.0759	0.2297	0.0427	0.0759	0.1258
2	0.0655	0.0859	0.2284	0.0819	0.0939	0.1205
3	0.0723	0.0869	0.2189	0.0597	0.0649	0.1315
4	0.0855	0.0958	0.2389	0.0432	0.1060	0.1295
5	0.0934	0.1339	0.2279	0.0357	0.1009	0.1318
6	0.0901	0.1369	0.2432	0.0811	0.1369	0.1445
7	0.0559	0.1640	0.2437	0.1302	0.1392	0.1518
8	0.1129	0.1820	0.2264	0.0829	0.1249	0.2006
9	0.0559	0.1730	0.2279	0.0829	0.1505	0.2910
10	0.1820	0.1820	0.2144	0.1189	0.1369	0.5658
Mean	**0.0876**	0.1316	0.2300	**0.0759**	0.1130	0.1993

Mean interval score ($\overline{S}_\gamma$)

	Female			Male		
h	α-transform	ilr	eda	α-transform	ilr	eda
1	0.0025	0.0031	0.0051	0.0033	0.0036	0.0024
2	0.0025	0.0031	0.0055	0.0035	0.0036	0.0029
3	0.0028	0.0032	0.0059	0.0037	0.0037	0.0036
4	0.0027	0.0031	0.0063	0.0035	0.0036	0.0041
5	0.0026	0.0032	0.0068	0.0035	0.0036	0.0047
6	0.0029	0.0035	0.0074	0.0037	0.0037	0.0055
7	0.0025	0.0037	0.0079	0.0038	0.0039	0.0064
8	0.0030	0.0036	0.0089	0.0039	0.0040	0.0081
9	0.0028	0.0039	0.0093	0.0043	0.0043	0.0098
10	0.0029	0.0042	0.0103	0.0041	0.0041	0.0137
Mean	**0.0027**	0.0035	0.0073	**0.0037**	0.0038	0.0061

Future research

1. A robust α transformation

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2. Extend α transformation to model & forecast life-table death counts for multiple populations

Future research

1. A robust α transformation
2. Extend α transformation to model & forecast life-table death counts for multiple populations
3. Consider cohort life-table death counts for specific groups of individuals

Thank you

Paper: `https://arxiv.org/abs/2305.19749`